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ON THE WRONSKIAN TEST FOR LINEAR DEPENDENCE.

BY MAXIME BÔCHER.

It is well known that the identical vanishing of the Wronskian, while a necessary condition for the linear dependence of functions of one variable* for which the necessary derivatives exist, is not a sufficient condition. It is, however, true that:

I. If $u_1(x), \cdots u_n(x)$ are throughout (a, b) analytic functions whose Wronskian $W(u_1, \cdots u_n)$ vanishes identically, then $u_1, \cdots u_n$ are linearly dependent throughout (a, b).

II. If $u_1(x), \cdots u_n(x)$ satisfy a homogeneous linear differential equation

$$\frac{d^k u}{dx^k} + p_1 \frac{d^{k-1}u}{dx^{k-1}} + \cdots + p_k u = 0 \tag{1}$$

at every point of an interval (a, b) throughout which $p_1, \cdots p_k$ are continuous, and if $W(u_1, \cdots u_n) \equiv 0$ in (a, b), then $u_1, \cdots u_n$ are linearly dependent in (a, b).

These two well-known theorems would probably be very nearly (if not quite) sufficient for all applications that have so far been made.

There are, however, many other cases which have been discovered by Peano, the writer,† and D. R. Curtiss,‡ in which the identical vanishing of the Wronskian implies linear dependence. The starting point for all these investigations is the following:

THEOREM A.§ *If $u_1, \cdots u_n$ have at each point of (a, b) finite derivatives of the first $n - 1$ orders, and if among the Wronskians of these functions taken $n - 1$ at a time there is at least one which does not vanish anywhere in (a, b), then, whenever $W(u_1, \cdots u_n) \equiv 0$, $u_1, \cdots u_n$ are linearly dependent throughout (a, b).*

For several years past it has been my custom in my lectures to establish first Theorem A and to deduce from this the two results I and II, using for this purpose an intermediate theorem. As I have not seen this theorem in the literature, and as I have recently found it useful for its own sake (not merely as a step towards I and II), I venture to give it here in spite of its rather obvious character.

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* Throughout this note we assume x real and speak of an interval (a, b). Everything said applies, however, equally to the case in which x is complex and we have to deal with a continuum in the complex plane. In this latter case, however, I is the *only* result worth recording.

† Trans. Amer. Math. Soc., vol. 2 (1901), p. 139, where references to earlier work by Peano will be found.

‡ Math. Ann., vol. 65 (1908), p. 282.

§ For a simple proof of this theorem see page 140 of my paper just cited.

THEOREM B. *If* $u_1, \cdots u_n$ *have finite derivatives of the first* $n-1$ *orders at every point of* (a, b), *then* $W(u_1, \cdots u_n) \equiv 0$ *is a necessary and sufficient condition that* $u_1, \cdots u_n$ *be linearly dependent throughout some sub-interval** *of* (a, b).

This is obviously (though trivially) true for a single function. We therefore assume that it is true for $n-1$ functions, and use mathematical induction.

Suppose $W(u_1, \cdots u_n) \equiv 0$. Then either

(a) $W(u_1, \cdots u_{n-1}) \equiv 0$, in which case $u_1, \cdots u_{n-1}$ are linearly dependent throughout some sub-interval, and hence the same is true of $u_1, \cdots u_n$; or

(b) there is a point of (a, b) where $W(u_1, \cdots u_{n-1}) \neq 0$, and hence, on account of the continuity of the derivatives of the first $n-2$ orders of the u's, there is a neighborhood of this point at no point of which does $W(u_1, \cdots u_{n-1})$ vanish. Hence in this sub-interval $u_1, \cdots u_n$ are linearly dependent by A. Thus B is proved.

The deduction of I and II from B is as follows:

If $W(u_1, \cdots u_n) \equiv 0$, we know from B that there exist constants $c_1, \cdots c_n$, not all zero, such that throughout a certain sub-interval (a', b')

$$c_1 u_1 + \cdots + c_n u_n \equiv 0. \tag{2}$$

If $u_1, \cdots u_n$ are analytic throughout (a, b), the same is true of the first member of (2). Hence, from the fact that this function vanishes throughout (a', b'), we infer that it vanishes throughout (a, b). Thus I is established.

On the other hand, if $u_1, \cdots u_n$ satisfy (1) at every point of (a, b), the same is true of the first member of (2), which we will call u. But, from (2), we see that u and its derivatives of all orders vanish at any interior point of (a', b'). This, by a fundamental theorem concerning equations of the form (1), implies that u is identically zero throughout (a, b). Thus II is proved.

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* And hence throughout some sub-interval of every sub-interval of (a, b).

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